November 22nd 2017 **Name (Please Print)** ______________________

Probability III- Final Exam - **Maximum Time: 3 hours** **Maximum Score: 100**

Instructions:

(a) This is closed book exam.

(b) If you are using any result stated/proved in class then please state the result and verify that the hypotheses are satisfied before using it.

1. You are waiting at Jayram Das bus stop.There are two routes that pass through the bus stop. Route no. 222A arrives in the form a Poisson process of rate one bus per hour and arrivals of Route no. 226 bus form an independent Poisson process of rate seven buses per hour.
   (a) (5 points) What is the probability that exactly three buses pass by in one hour?
   (b) (5 points) What is the probability that exactly three 226 buses pass by while you are waiting for a 222A?
   (c) (5 points) When the Kengeri maintenance depot goes on strike half the buses break down before they reach Jayram Das bus stop. What, then, is the probability that you wait for 30 minutes without seeing a single bus?

2. Let $\{X_t\}_{t\geq 0}$ be a right continuous process on $\{0,1\}$. Let $\{Y_n\}_{n\geq 0}$ be the jump-chain associated to $\{X\}_{t\geq 0}$. Suppose the holding times $\{S_n\}_{n\geq 1}$ are independent random variables exponentially distributed with parameters λ_n given by

$$\lambda_n = \begin{cases} \mu & \text{if } Y_n = 1 \\ \lambda & \text{if } Y_n = 0 \end{cases}$$

   (a) (5 points) Describe the generator matrix, Q, associated with $\{X_t\}_{t\geq 0}$.
   (b) (10 points) Suppose the initial distribution X_0 is given by (π_0, π_1). Find $P(X_t = 1)$.

3. Consider the Markov chain on $\{1,2,3,4\}$ with generator matrix

$$Q = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{6} & 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   For $i \in S$, let $T^{\{i\}} = \inf\{t \geq 0 : X_t = i\}$.
   (a) (5 points) Find the jump matrix π associated with Q.
   (b) (10 points) Find $P(T^{\{3\}} < \infty \mid X_0 = 1)$.
   (c) (10 points) Find $E(T^{\{4\}} \mid X_0 = 1)$.

4. Let X_n be a Markov chain on $S = \mathbb{Z}$ with tranistion matrix P given by

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

   where $0 < p < 1$ for all $i \in \mathbb{Z}$.

(a) (5 points) Show that the chain is irreducible and determine its periodicity.

(b) (10 points) For each $0 < p < 1$, decide whether the chain is null recurrent, positive recurrent or transient.

5. Consider the discrete time Markov chain on $\{1, 2, 3\}$ with transition matrix P

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Calculate

(a) (5 points) Find p_{11}^n for all $n \geq 1$.

(b) (10 points) Let $N_n(i)$ be the number of visits to site i until time n. For $i = 1, 2, 3$ decide if

$$\lim_{n\to\infty} \frac{N_n(i)}{n},$$

exists with Probability one.

6. Let X_n be a Markov chain on state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

(a) (5 points) Sketch the graph induced by this Markov chain on the vertex set S.

(b) (10 points) Determine the closed communicating classes and their periodicity.